ISRAEL JOURNAL OF MATHEMATICS, Vol. 39, No. 3, 1981

NOTE ON PRIMITIVE WORDS

BY

E. RIPS

ABSTRACT

This note presents an example that disproves, for $n = 4$, Weinbaum's conjecture, that if w is a cyclically reduced primitive word in F_n such that all the generators $x \in X$ appear in w then some cyclic permutation of w can be partitioned into n words generating $F_n : w \equiv uv$, $vu \equiv s_1 s_2 \cdots s_n$, $\langle s_1, s_2, \cdots, s_n \rangle = F_n$.

An element of a free group is called *primitive* if it can be included in a set of free generators of the group.

Let F_n be a free group of rank n on a fixed set X of free generators. We identify the elements of F_n with reduced words. Weinbaum [3] conjectured that if w is a cyclically reduced primitive word in F_n such that all the generators $x \in X$ appear in w then some cyclic permutation of w can be partitioned into n words generating $F_n : w \equiv uv$, $vu \equiv s_1 s_2 \cdots s_n$, $\langle s_1, s_2, \cdots, s_n \rangle = F_n$.

For $n = 2$ this was proved by Nielsen [2], and for $n = 3$ and a positive w the same was proved by Weinbaum [3].

We bring an example that disproves this conjecture for $n = 4$. Let $X = \{a, b, c, d\}$. The element

$$w \equiv b^{-1}a^{-1}bab^{-1}abd^{-1}c^{-1}dcd^{-1}cd$$

is cyclically reduced, contains all the generators and it is primitive because it belongs to the following system of free generators of F_4:

$$\{b^{-1}a^{-1}bab^{-1}abd^{-1}c^{-1}dcd^{-1}cd, b^{-1}a^{-1}bab^{-1}ab,$$

$$b^{-1}a^{-1}baba^{-1}b^{-1}ab, d^{-1}c^{-1}dcdc^{-1}d^{-1}cd\}.$$

We claim that no cyclic permutation of w can be partitioned into 4 words generating F_4.

The following argument enables us to avoid checking all the possible cases.

Received January 2, 1981

Let $u \equiv b^{-1}a^{-1}bab^{-1}ab$ and $v \equiv d^{-1}c^{-1}dcd^{-1}cd$. It is easy to check that u (v) cannot be partitioned into two words that generate $\langle a, b\rangle$ ($\langle c, d\rangle$). If the *cyclic* word $w \equiv uv$ is partitioned into words that generate F_4 then intersections of these words with u (v) generate $\langle a, b\rangle$ ($\langle c, d\rangle$). Therefore, each of the words u and v must have a non-empty intersection with at least three of these words. In particular, if there is a partition of the cyclic word uv into four words generating F_4 then $u \equiv u_0u_1u_2$, $v \equiv v_0v_1v_2$ and

$$\langle u_1, u_2v_0, v_1, v_2u_0\rangle = F_4$$

where $u_0, v_1, u_2, v_0, v_1, v_2$ are non-empty words.

There are 15 different ways to partition u into 3 non-empty words. In the following 3 cases

(1) $u_0 \equiv b^{-1}$, $u_1 \equiv a^{-1}bab^{-1}a$, $u_2 \equiv b$;

(2) $u_0 \equiv b^{-1}a^{-1}$, $u_1 \equiv bab^{-1}$, $u_2 \equiv ab$;

(3) $u_0 \equiv b^{-1}a^{-1}b$, $u_1 \equiv a$, $u_2 \equiv b^{-1}ab$

the words u_0, u_1, u_2 do not generate $\langle a, b\rangle$. We collect the 12 remaining possibilities for u_0, u_1, u_2 and 12 similar possibilities for v_0, v_1, v_2 into the following table:

	u_0	u_1	u_2	v_0	v_1	v_2
1	b^{-1}	a^{-1}	$bab^{-1}ab$	d^{-1}	c^{-1}	$dcd^{-1}cd$
2	b^{-1}	$a^{-1}b$	$ab^{-1}ab$	d^{-1}	$c^{-1}d$	$cd^{-1}cd$
3	b^{-1}	$a^{-1}ba$	$b^{-1}ab$	d^{-1}	$c^{-1}dc$	$d^{-1}cd$
4	b^{-1}	$a^{-1}bab^{-1}$	ab	d^{-1}	$c^{-1}dcd^{-1}$	cd
5	$b^{-1}a^{-1}$	b	$ab^{-1}ab$	$d^{-1}c^{-1}$	d	$cd^{-1}cd$
6	$b^{-1}a^{-1}$	ba	$b^{-1}ab$	$d^{-1}c^{-1}$	dc	$d^{-1}cd$
7	$b^{-1}a^{-1}$	$bab^{-1}a$	b	$d^{-1}c^{-1}$	$dcd^{-1}c$	d
8	$b^{-1}a^{-1}b$	ab^{-1}	ab	$d^{-1}c^{-1}d$	cd^{-1}	cd
9	$b^{-1}a^{-1}b$	$ab^{-1}a$	b	$d^{-1}c^{-1}d$	$cd^{-1}c$	d
10	$b^{-1}a^{-1}ba$	b^{-1}	ab	$d^{-1}c^{-1}dc$	d^{-1}	cd
11	$b^{-1}a^{-1}ba$	$b^{-1}a$	b	$d^{-1}c^{-1}dc$	$d^{-1}c$	d
12	$b^{-1}a^{-1}bab^{-1}$	a	b	$d^{-1}c^{-1}dcd^{-1}$	c	d

In 136 cases out of 144 (excluding the cases 1–7, 1–12, 4–7, 4–12, 7–1, 7–4, 12–1, 12–4) the set $\{u_1, u_2v_0, v_1, v_2u_0\}$ has the Nielsen property and therefore does not generate F_4. In the 8 remaining cases we get a set with the Nielsen property after doing one elementary transformation and in such a way we check that the set $\{u_1, u_2v_0, v_1, v_2u_0\}$ does not generate F_4.

For example, in the case 1–7, applying an elementary transformation to the set $\{a^{-1}, bab^{-1}abd^{-1}c^{-1}, dcd^{-1}c, db^{-1}\}$, we obtain a new set $\{a^{-1}, bab^{-1}abd^{-1}c^{-1}, bcd^{-1}c, db^{-1}\}$ which possesses the Nielsen property. Hence it does not generate F_4 and then $\langle a^{-1}, bab^{-1}abd^{-1}c^{-1}, dcd^{-1}c, db^{-1}\rangle \neq F_4$.

References

1. W. Magnus, A. Karrass and D. Solitar, *Combinatorial Group Theory*, Wiley, 1966.

2. J. Nielsen, *Die Isomorphismen der allgemeinen, unendlichen Gruppe mit zwei Errengenden*, Math. Ann. **78** (1918), 385–397.

3. C. M. Weinbaum, *Partitioning a primitive word into a generating set*, Math. Ann. **181** (1969), 157–162.

Institute of Mathematics
The Hebrew University of Jerusalem
Jerusalem, Israel